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THE CONCEPT OF GROUP AND THE THEORY OF PERCEPTION*

I

The first attempt to apply certain mathematical speculations concerning the *concept of group* to psychological problems of perception was made by Helmholtz in his essay *Ueber die Tatsachen, die der Geometrie zu Grunde liegen* (1868). To be sure, Helmholtz was not able to see the new problem which he had raised with complete precision and to realize its full importance. For, when Helmholtz wrote his essay, the concept of group was not yet recognized as that universal instrument of mathematical thought which it later turned out to be. Its application was confined to certain problems of combinatorics and algebra. At the beginning of the nineteenth century, Cauchy had introduced the concept of group into these domains. In Galois' theory of algebraic equations the concept had proved eminently fruitful. It was, however, not before the second half of the century that those studies were inaugurated through which the theory of groups was to be established as a special discipline. Very soon the theory found applications in the most varied branches of mathematics as an organizing and clarifying principle.[1] Its establishment is closely connected with a general reorientation of geometrical thought. The new orientation, brought about by the discovery of the non-Euclidean geometries, succeeded in a fully satisfactory way, when Sophus Lie and Felix Klein assigned to the concept of group a central position in the system of geometrical thought. Helmholtz could not avail himself of all these advances. He therefore fails to give an explicit definition and analytical clarification of the concept of group. Nonetheless his essay contains a great many promising concrete starting-points and problems. For some errors of analytical treatment which were disclosed and corrected later, the reader is more than compensated by the breadth of Helmholtz' general epistimological horizon and the vigor of synthesis which enables him to bring together problems pertaining to highly different fields of study.

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[1] For details cf. Felix Klein, *Vorlesungen ueber die Entwicklung der Mathematik im XIX. Jahrhundert*, Part I, Berlin, 1926, pp. 334 ff.

Helmholtz avowed that it was his dealing with the fundamental problems of "Physiological Optics" that encouraged and, in a certain sense, even enabled him to undertake this synthesis. From the outset his attention was drawn to the question as to whether and to what extent experience contributes toward shaping the notion of space. He was a Kantian in so far as he endorsed the thesis of space as a "transcendental form of intuition," and he persistently clung to this thesis. But this thesis was to him the beginning, and not the solution, of the problem. According to Helmholtz, the transcendental form merely designates the general "possibility of coexistence"—as space had been defined by Kant. As soon as we attempt to specify this possibility—and only through such specification can it be made applicable to and fruitful for the problems of physics—we find ourselves faced with a whole new set of questions. We must now introduce a metrical determination. In contradistinction to the general form of space as such, this determination is not given *a priori;* it may be introduced in different ways. All concrete measurement depends upon the acceptance of certain axioms of congruency between different parts of space. The examination of these axioms shows that they imply certain presuppositions as to the extent to which figures may be displaced without transformation. Thus Helmholtz tackles the problem of finding the most general form of a multidimensional manifold in which rigid bodies or systems of points may be displaced relative to one another without changing their forms. The axioms at the basis of every geometry may then be interpreted as statements concerning determinate groups of movement. The objective validity of these axioms depends not merely upon the *a priori* "form" of space, but upon fundamental experiments performed on "rigid bodies." It appears that in a three-dimensional space of constant curvature the possible displacements depend upon six parameters. The motions of three-dimensional space are ∞^6, and form a group, say G_0. This group is known to have an invariant; but the form of this invariant in terms of the coordinates x_1, x_2, x_3, y_1, y_2, y_3 of the points is not known *a priori*. The question arises whether the group of motions is fully characterized by these two properties, so that none but the Euclidean and the two non-Euclidean systems of geometry are possible. There is then here a sextuple infinity of possible movements. Their study provides us with the most illustrative example of what later was quite generally called "group of transformations." Lie and Klein define the group as the totality of unique operations $A, B, C \cdots$ so that from the combination of any two operations A and B there results an operation C which also belongs to the totality: $A \cdot B = C$. The generalization of geometry leads to the following problem: "Given a multiplicity and a group of transformations referring to the former; the problem is to study the elements of the multiplicity with regard to those properties which are not affected by the transformations of the group."[2] Every system of geometry is characterized by its group: It deals only with such relations of space as remain unchanged through the transformations of its group.

It is from the point of view of this fundamental conception that Henri Poincaré tackles the problem of space and perception of space. But according to Poincaré the relation between conception and perception is different from what it is in Helmholtz' "empiricistic" doctrine. After the concept of space had been defined by, and even in a certain sense reduced to, the concept of group, the epistemological solution of the "Helmholtz-Riemann" problem had to start from this point. The logical nature of the concept of group had to be formulated in detail and established. In this respect it is impossible to resort simply to "experience." In fact, the theory of group, as Hermann Weyl says,[3] is the most striking example of "pure intellectual mathematics." To understand and logically justify this theory, we must, according to Poincaré, turn to an original law 'of the human mind' and not to the nature of "external things". Poincaré does not hesitate to recognize the concept of group as a true fundamental concept *a priori*. This concept derives from an original "intuition" which precedes and underlies all experience, just as that other intuition to which Poincaré traces the construction of the series of the natural numbers, and also the principle of "mathematical induction." The object of geometry is the study of a particular "group" of transformations; the general group-concept, however, "preexists" in our minds, at least potentially. It is, as Leibniz would say, a concept of *intellectus ipse*: "it is imposed on us not as a form of our sensibility, but as a form of our understanding."[4] Guided by this insight into the fundamental logical importance of the concept of group, Poincaré traces the limits of every empiricistic explanation of geometry. There is an irreducible difference between axioms of geometry and empirical statements derived from observation and measurement. The two cannot be directly compared since they belong to entirely different orders of objects. "We do not make experiments on ideal lines or ideal circles; we can only make them on material objects."[5] Statements concerning the latter can never validate or invalidate the former. Their

[2] Felix Klein, "Vergleichende Betrachtungen ueber neuere geometrische Forschungen," *Erlanger Programm*, 1872; cf. *Gesammelte mathematische Abhandlungen*, Berlin 1921, vol. I, p. 461. See also F. Klein, The Evanston Colloquium, Lectures on Mathematics, New York and London, 1894, Lecture XI: The Most Recent Researches in Non-Euclidean Geometry, pp. 85 ff.

[3] "Philosophie der Mathematik und Naturwissenschaft" in *Handbuch der Philosophie*, Munich and Berlin, 1926, II A, p. 23.

[4] Poincaré, *La Science et l'Hypothèse*, p. 90. English translation, *The Foundations of Science*, by George Bruce Halsted, New York, The Science Press, 1913, pp. 79 f.

[5] *Ibid.*, p. 65 (English translation p. 64).

validity is that of the creative mathematical definition, which is restricted by no other rule than that of avoiding contradictions. As to the three geometries of constant curvature—the geometries of Euclid, Lobatschefsky, and Riemann—none of them may be invalidated by experience. All that experience can do is lead the mind in a certain direction as a result of which it may construct such a system of geometrical concepts as yields the simplest and most convenient instrument for the description of physical phenomena. "In our mind the latent idea of certain number of groups preexisted . . . . Which shall we choose to form a kind of standard by which to compare natural phenomena? And when this group is chosen, which of the sub-groups shall we take to characterize a point in space? Experience has guided us by showing us what choice adapts itself best to the properties of our body; but there its role ends."[6]

Poincaré resorts to the concept of group for still more concrete problems. Studying infinite groups, Lie found it necessary to postulate that besides an operation A the inverse operation A^{-1} must also be present in the group. He had brought into his *definition* of the group the requirement of the presence of the inverse transformation along with every admitted transformation.[7] Poincaré starts from this mathematical fact and relates it in an original way to a psychological problem. What the perceiving subject immediately experiences is an almost uninterrupted flux of sense impressions. How, in the face of this fact, is that differentiation possible which we constantly make in our interpretation of these impressions, viz., the differentiation between spatial movements of an object and its qualitative alterations? The mere psychological clues are the same in both cases. Only by the alteration in the perceptual images are we informed of a change, whether the latter consist in that the object is removed from our bodily organs or in a modification of the object itself. We must then find another criterion which permits us to discriminate between the two cases. In fact, in the one case, when the object has merely been displaced, we are able to restore the original perception by making movements so as to put the object again in that position relative to our body in which it had been before it was displaced. What characterizes displacement and distinguishes it from qualitative modification is, from the psychological point of view, nothing else but this possibility of correction and "compensation." How is such a compensation *possible?* How does it come about that two successive and independent changes neutralize each other and lead back to the same initial state? This question cannot be answered with true exactness until the elaboration of geometry has been completed and based upon certain definitions of group-theory. Experience can only tell us that the correction does,

[6] *Ibid.*, pp. 87 f.

[7] Cf. Maurer and Burckhardt, "Kontinuierliche Transformationsgruppen," *Enzyklopaedie der Mathematik*, II A6; vol. II, Part I, p. 402 (English edition).

as a matter of fact, occur; thus experience may offer the occasion to create the geometrical concepts required for the intellectual representation of the fact.[8] Here again experience proves not to be the source of concepts, but merely the occasional cause of their formation.

That this combination of ideas is original and stimulating in its originality will readily be conceded. But both psychologists and mathematicians will refuse to take a further step and to allow that Poincaré was here formulating a genuine *fundamental problem of methodology* with which both mathematics and psychology must deal, although from different sides. What we have expounded seems to be one of those ingenious aperçus characteristic of Poincaré, the thinker and the writer. But I am convinced that the present state of the psychology of perception compels us to hold a different opinion. In the following reflections I shall attempt to set forth an inner connection—epistemological in nature—between the mathematical concept of group and certain fundamental problems of the psychology of perception as the latter have been more and more distinctly formulated in the last decades. To this end, we must look far afield. For the two scientific provinces which we are trying to connect appear at first sight to be entirely disparate as to their content. Yet, we should not allow ourselves to be misled by this disparity. What we are going to set forth concerns logic only, and not ontology. Our ultimate aim is to bring out clearly a certain *type of concepts* which has found its clearest expression in abstract creations of modern geometry. But the type in question is not confined to the geometrical domain. It is, on the contrary, of far more general validity and use. The application of concepts of this type extends both farther and deeper. Metaphorically speaking, it extends down to the very roots of perception itself. Perception too cannot be understood in its specific nature, meaning, and total structure without the assumption of organization, coordination, and synthesis. "The process of our comprehension with respect to natural phenomena"—thus Helmholtz defines his general problem in his *Treatise on Physiological Optics*—"is that we try to find *generic notions* and *laws of nature.*" Laws of nature are merely generic notions for the changes in nature. . . .When we cannot trace natural phenomena to a law, and therefore cannot make the law objectively responsible as being the cause of the phenomena, the very possibility of comprehending such phenomena ceases. However, we must try to comprehend them. There is no other way of bringing them under the control of our intellect. And so in investigating them we must proceed on the supposition that they are comprehensible. Accordingly, the law of sufficient reason is really nothing more than the *urge* of our intellect to bring all our perceptions under its own control."[9]

[8] Poincaré, *La Science et l'Hypothèse*, pp. 74 ff., English translation, pp. 70 ff.

[9] Helmholtz, *Handbuch der Physiologischen Optik*, 2nd edition, 1896, pp. 591 f. English translation by James P. C. Southall, 1925, vol. III, p. 34.

This "comprehension of the phenomenon by thought" is the common task of all knowledge and—as we shall try to show—the intermediary link between the logical system of geometrical concepts and the phenomenology of sense-perception.

II

In "*Vergleichende Betrachtungen ueber neuere geometrische Forschungen,*" in which he laid down the program of modern geometry, Felix Klein postulates that first of all the concept of a geometrical property of an object shall be defined in exact terms. Not every apprehension and description of a spatial object is, by this token, a geometrical characterization. If we consider the object simply in its *hic et nunc*, looking merely at its individuality, the latter does not reveal its geometrical character and significance. By describing a spatial form as such in its particularity and concreteness, we attain, at the utmost, to its geographical or "topographical," but not to its "geometrical" concept. To establish the latter, a new and quite different direction of thought is required. As Klein formulates the new princple: "The geometrical properties of any figures must be describable in terms of formulae which do not change when the system of coordinates is changed; conversely, any formula, which in this sense is invariant with respect to the group of given transformations of the coordinates, represents a geometrical property." As the most important transformations of this kind we may consider parallel displacement, rotation through a definite angle, symmetry with regard to the x-axis, and alteration of the scale. So far as Euclidean geometry is concerned, it is characterized and distinguished from other geometries which logically are equally possible and equally justified by the fact that it considers a *principal group* of spatial relationships and investigates the invariant properties with respect to this group. The group in question consists of a sextuple infinity of movements, a uni-dimensional infinity of transformations by similarity, and the transformation by reflexion in the plane. Geometry deals only with those properties of spatial figures which are independent of the location of the figures and also of their absolute magnitude; it does not distinguish between the properties of a body and those of its image produced by a mirror.[10]

From this definition of "geometrical properties" the conditions become immediately apparent under which two spatial concepts are "equivalent" to each other, i.e., are but different expressions of one and the same geomet-

[10] Felix Klein, "Erlanger Programm von 1872," *Gesammelte Mathematische Abhandlungen*, vol. I, pp. 46 ff. Cf. especially *Mathematische Annalen*, vol. VI; 1873 (*Ges. Abhandl.*, vol. I, p. 315 ff.). See also the development of the principal ideas in Klein's work *Elementarmathematik vom hoeheren Standpunkt aus*, 3rd edition, Berlin, 1925; vol. II, pp. 27 f.

rical "essence." The "essence" of a triangle is not altered, the logical *assertions* about it are not invalidated, when we change its individuality in certain ways, e.g., displace it in space or make the absolute lengths of the sides increase or decrease. We may say quite generally that two series of expressions which are transformed in this manner must be considered as geometrically equivalent, i.e., defining identical geometrical figures. To see the full significance and methodological fruitfulness of this definition, we have but to bear in mind that in the choice of the group of transformations we are entirely free and not confined to any preconceived scheme. For it appears that every change of the system of reference entails a change as to that which we have to consider as a geometrical property and as equivalent figures. According to the modern conception advocated by Klein, the characteristic properties of a multiplicity must not be defined in terms of the *elements* of which the multiplicity is composed, but solely in terms of the *group* to which the multiplicity is related. As soon as we substitute one group for another there result, therefore, quite different correspondences. What had appeared as expressions of the "same" geometrical concept may be separated; what had appeared to be specifically different may turn out to be generically identical. This becomes most apparent in the transition from metrical to *projective geometry*. That the latter, in comparison with the classical form of metrical geometry as represented by Euclid, is wider and more general, became more and more evident in the course of the development of projective geometry, introduced by Poncelet and furthered by Moebius, von Staudt, and Caylay. "Metrical geometry is a part of descriptive geometry, and descriptive geometry is all geometry and reciprocally," declares Caylay. From the standpoint of group-theory this conception is an immediate consequence. The group at the base of projective geometry is wider than that underlying metrical Euclidean geometry, since to the transformations by similarity in the usual sense there are adjoined parallel and central projections and all transformations derived from the latter.[11] Says Klein· "Projective geometry developed only when one begun to consider the original form and all those forms resulting from the latter by projection as essentially identical and to formulate the properties transferred by projection so as to make appear their independence from the alteration connected with projection." Thus, what in the geometrical sense must be taken as "identical" and what as "different" is by no means predetermined at the outset. On the contrary, it is decided by the nature of the geometrical investigation, viz., the choice of a determinate group of transformations. From the standpoint of metrical Euclidean geometry, e.g., the different conics appear as distinct entities, as independent geometrical individualities

[11] Cf. H. Weyl, *loc. cit.*, p. 59.

which have definite and well-defined properties. This distinction disappears when the point of view is changed. If we allow for the so-called "affinitive transformations," we can no longer maintain the distinction between "circle" and "ellipse" in the traditional sense, since by affinitive transformation circles are transformed into ellipses. This development is carried still farther in projective geometry in which quite generally an ellipse may be transformed into a parabola or a hyperbola, such that, in the final analysis, there is but one single conic. It appears from all this that the concepts of modern geometry derive their precision and true universality only from the fact that the intuited particular figures are not considered as pre-given and rigid, but rather as a kind of plastic material capable of being moulded into the most varied forms. The real foundation of mathematical certainty lies no longer in the elements from which mathematics starts but in the *rule* by which the elements are related to each other and reduced to a "unity of thought."

The progress achieved in the construction of the universe of geometrical concepts may be illustrated from another side. The transition from mere "topographical" to genuinely geometrical properties may be characterized as a progress from merely *local* to truy *spatial* determinations. All those determinations that can be given only by "pointing," a τόδε τι in Aristotle's sense, are "local." These determinations refer to a simple *hic et nunc* which can only be pointed at; their meaning derives from a concrete intuitive situation. From this viewpoint all individual differences between figures are equal in value and importance. Every particular triangle, every particular circle is to be considered as something in and by itself. Its location in space, the lengths of the sides of the triangle or of the radii, etc., belong to its "nature," which latter cannot be defined except with reference to particular local circumstances. Even in our geometrical concepts, this reference is not simply ignored; it is not abstracted from such as to simply disappear. But here the local determinations are comprehended in such a way that a new whole, the "system of space," results from their synthesis. The concept of the group of transformations is, perhaps, the clearest expression of the nature and epistemological root of this systemitization. Owing to this concept the particular, intuitively given figure is deprived of its *hic et nunc* and nevertheless retains its definiteness. This definiteness no longer depends upon what the figure is as a "this" or "that," as a particular. The definiteness of the figure depends upon the context into which it is integrated and which it represents as a special case. The more we enlarge this context by broadening the "principal group" of spatial transformations which we started out from, and by successively "adjoining" groups of transformations each of which contains the preceding ones, the more we approximate the genuinely universal system of space, the aim of geometrical conceptualization.

III

What we discussed last seems to digress far from the problems of the *perceptual world*. It is characteristic of perception that it can never attain to that stage which represents the beginning of geometrical thought. Perception cannot abandon the *hic et nunc*, since its peculiar task is just to apprehend the *hic et nunc* as precisely and completely as possible. If perception would cease to have an *individual* content, it would cease to have any content whatsoever. We do not deny the possibility of perceptual content. We do not deny, for rationalistic or intellectualistic reasons, that perception never will and never can attain to that form of universality for which geometrical thought is striving. On the other hand, the sensualistic thesis, which modern psychology started out from, cannot be maintained either. It is not the force of epistemological objections, but a simple clarification of the *phenomenological* facts involved in perception, that has refuted this thesis. When nowadays one attempts to describe these facts as they are revealed by experiment and precise analysis, one can no longer stick to the conception that perception is nothing but a bundle of sense-impressions. That the perceptual world does possess a *structure* and that this structure cannot be reduced to a mere mosaic, an aggregate of scattered "sensations," may be taken as an established conclusion of psychology, and it is upon this conclusion that we base our reflections in the following.

We cannot, for our purpose, content ourselves with a *general* formulation of this idea. We must pursue it into details, into concrete facts. First of all, we are confronted with the fact of *perceptual constancy*. Since *Helmholtz'* "Physiological Optics," since *Hering's* fundamental investigations into the sense of light, the phenomenon in question has been set forth with more and more clarity. The greater the clarity with which it was brought out, the more definite became the epistemological *problem* that is involved in this phenomenon. Gelb writes: "In general, when a sheet of paper appears white in ordinary daylight, we do not hestiate to recognize it as white in very dim light as well, e.g., in the light of the full moon; and a piece of velvet which looks black to us under a cloudy sky, looks also "black" to us in full sunshine. The same sheet of paper appears white also in the greenish shadow of foliage; and so it appears in the rays of one or the other of the usual artificial sources of light, all of which emit more or less chromatic light. Similar observations may eventually be made on *colored* objects, although to a lesser extent and with less clarity; a piece of paper, e.g., which looks blue in daylight looks blue also in the reddish-yellow light of a gas-flame. Observations of this kind show that considerable changes of the *intensity* of illumination and, within certain limits, also of the *color* of illumination do not, in any appreciable degree, influence our ordinary vision of colors. *This fact becomes a problem* when we consider that every change of illumination entails a change in the radiation which the external objects reflect on our eyes, so that every change of illumination is accompanied by a modification in the stimulation of the retina." This problem of the "approximate color constancy of visible things" (*angenaeherte Farbenkonstanz der Sehdinge*), as Hering called it, is not unique. Besides the color constancy there is constancy of spatial shape and size. When an object is moved away from our eyes, the images on the retinae become smaller and smaller. Nonetheless, within certain distances, the perceptual size of the object is constant. Variations of shape, which result from the fact that a figure is turned out of the frontal-parallel position, are also "counterbalanced" by the eye to a high degree, so that we perceive the figure in its "true" shape. *What is meant by this "truth"*—a kind of truth which seems to contradict the objective facts, the real conditions of physical stimulation? In raising this question, psychological inquiry comes close to the fundamental epistemological problems of the theory of perception, even though it may try to confine itself strictly to empirical observation. It is of great interest to study this development of concepts and methods, following the excellent critical report given by A. Gelb of the origin and development of the problem of "color constancy of visible things."[12] As to the *explanations* of the phenomenon, they cannot, so far as I can see, be reduced to a single formula. The theories advanced by Helmholtz, Hering, Joh. v. Kries, Katz, Buehler, Jaensch, and others diverge on essential points. But there seems to be complete agreement as to the phenomenal fact itself and its *significance*. The phenomenon under discussion evidently arouses the philosophic "wonder" of psychologists more than any other phenomenon. Buehler says that the color constancy of visible things, the approximate invariance of the qualitative black-white series with respect to changes of illumination, must be reckoned among "the most astonishing perceptual achievements of the eye."[13] He emphasizes that the law in question is of decisive importance for the knowledge and recognition of visible things and hence for the possibility of intelligent human and animal behavior, as far as the optical sector is concerned.[14] Katz maintains that phenomena analogous to those in the optical domain may be observed in nearly all other domains of perception. "The idea of invariance, which is an epistemoligical problem of validity of the foremost importance, has one of its roots, and perhaps the most nutritive one, in the psychology of perception."[15] Gelb concludes his critical report with the

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[12] A. Gelb, "Die Farbenkonstanz der Sehdinge," in *Handbuch der normalen und pathologischen Physiologie*, edited by Bethe, vol. XII, pp. 594–678.

[13] K. Buehler, *Handbuch der Psychologie*, I, 1: Die Erscheinungsweisen der Farben, Jena, 1922, pp. 73 f.

[14] Buehler, *Die Krisis der Psychologie*, Jena, 1927, p. 71.

[15] Katz, *Der Aufbau der Farbwelt*, Leipzig, 1930, p. 300.

statement that color constancy is but a part of a much more complex set of problems; we are confronted with the general problem of the organization and structure of the visible world.[16]

We cannot dwell here upon the psychological facts themselves and their phenomenological analysis,[17] nor can we concern ourselves with the general epistomological consequences of these facts.[18] I content myself with setting forth that particular feature which seems to be most important with regard to our ealier consideration. If one surveys the facts as they have been described by psychologists, one meets again and again with two fundamental concepts that are familiar to us from another trend of thought: the concepts of "invariance" and "transformation." Both Helmholtz and Hering had emphasized that the objective stimuli are not simply "copied" in perception, but "transformed" in a certain direction, although they disagreed in their interpretations of the factor conditioning and determining this transformation. Helmholtz resorted to a function of *judgment*, Hering to a function of *memory*, in addition to certain phsyiological facts like pupilary variations and the mutual interaction of the elements of the visual field. Neither theory, however, gives a complete account and an exhaustive interpretation of the phenomenal facts.[19] The causes of this "transformation" must be sought for elsewhere. The phenomenon in question must be described in terms other than those that derive from the assumption that a given "sensation" is modified by intellectual and reproductive factors. The question which I should like to raise first is, whether it is merely by accident that a concept belonging to group theory appears in the very exposition of the psychological facts. One might think that the use of this term in a psychological context is ambiguous or merely metaphorical. One ought by no means to allow oneself to indulge in the illusion of "mathematical psychology," as such a discipline was tentatively developed by Herbart in a merely speculative way. The precision of mathematical concepts rests upon their being confined to a definite sphere. They cannot, without logical prejudice, be extended beyond that sphere into other domains.

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[16] Gelb, *loc. cit.*, p. 672.

[17] As regards these facts and the other particular phenomena on which the following discussion is based, I refer especially to the work of Katz: *Die Erscheinungsweisen der Farben und ihre Beeinflussung durch die individuelle Entwicklung*, 1st edition, Leipzig, 1911. In the second edition of Katz' work (*Der Aufbau der Farbwelt*, Leipzig, 1930) all of the relevant literature up to 1930 is critically mentioned and discussed. As to later discussions I refer especially to Egon Brunswik's *Wahrnehmung und Gegenstandswelt, Grundlegung einer Psychologie vom Gegenstand her*, Leipzig and Vienna, 1934, and to L. Kardos, *Ding und Schatten*, Leipzig, 1934.

[18] Cf. author's "*Philosophie der symbolischen Formen*," vol. III, 1929, p. 137 ff.

[19] Cf. Katz' criticism in *Der Aufbau der Farbwelt*, pp. 430 ff., and Buehler, *Handbuch der Psychologie*, I, pp. 114 ff., 124 ff.

While avoiding the error of such illegitimate extrapolation, we may nevertheless insist upon a *mediate* connection. The latter is revealed when we consider that form of "universality" which is the ultimate logical function of mathematical concepts but which, on the other hand, may also be present in the basic phenomena of perception that are usually described in the language of "sensationism." According to the sensationistic theory, the function and cognitive significance of perception consists in its close adaptation to the object, i.e., the bare stimuli. On the basis of this assumption, perception appears as immediate mechanical reproduction. Hobbes was the first one who clearly and explicitly expressed this view. What we call "perception" is, for Hobbes, nothing but an organism's reaction to external stimulation. "Action" and "reaction" can be related in no other way than strict equality. Hobbes anticipates, within the domain of psychology, Newton's mechanical law of the equality of "action" and "reaction." He goes so far as to *define* perception in terms of this law: *Sensio est ab organi sensorii conatu ad extra, qui generatur a conatu ab objecto versus interna, eoque aliquandiu manente per reactionem factum phantasma.*[20] Modern psychology developed on the basis of this assumption which, indeed, had to be modified in essential respects and was thus deprived of its classical "simplicity." Modern psychology rested on the "constancy-hypothesis," i.e., the hypothesis of an immediate correspondence between "stimulus" and "sensation." From the point of view of methodology, one of the most important results of Hering's inquiries into the "sense of light" consists in the explicit abandonment of this hypothesis. Hering raises the question with precision: "Do equal relations between light-intensities on the side of real things correspond to equal differences of brightness on the side of things as they are seen?" Hering's answer to this question is negative.[21] Thus the problem of "perceptual constancy" acquires a new meaning, and is so to speak, assigned its proper locus in the proper *dimension*. It henceforth appears that it is *dissimilarity* rather than *similarity* to the objective stimulus which characterizes perceptual content. This similarity may, indeed, be artificially produced and forced upon the perceptual process. But it can be actualized only under artificial experimental conditions that differ essentially from those of "normal" perception. By means of what Katz calls "reduction" we can transform the perceived "surface colors" into pure "film colors," viz., by looking through a hole in a screen. The "film colors" which then appear in the hole look quite different from that on surfaces of the two objects as they are directly observed. The former

[20] Hobbes, *De Corpore*, ch. 25, 2; cf. esp. *Leviathan* ch. 1.

[21] *Grundzuege der Lehre vom Lichtsinn*, 1905, p. 81; cf. L. Kardos, "Die 'Konstanz' phaenomenaler Dingmomente," *Festschrift zu Karl Buehler's 50. Geburtstag*, Jena, 1922, p. 22.

colors correspond to the conditions of physical stimulation in the sense that the film color which appears brighter and more closely approximates to whiteness is the one that is produced by more intensive physical radiation.[22] It is just the fact that in this way we do not stick to the given, the *hic et nunc*, the particular stimulus, and to the immediate impression produced in us by this stimulus, which constitutes the *real* problem of perception. We do not merely "*re*-act" to the stimulus, but in a certain sense act "against" it. In free perception the sensory material which is presented to us owing to optical stimulation is "dissociated by the inner eye."[23] Helmholtz already had explicitly pointed to this "dissociation," the permanent tendency to discriminate within the color or the visible appearance of an object what is due to the effects of illumination from what belongs intrinsically to the object. The essential conclusion hence to be drawn is that perception in general is not confined to the mere *hic et nunc*. Perception expands the particular datum; it is integrated into a total experience; and it is only in virtue of this integration that perception can exercise its proper function as an objective factor in knowledge. If perception were tied up with the flux of impressions, it would necessarily disintegrate; for each of these impressions presents the size, shape, and color of the object in a different way. As a matter of fact, however, perception does not stick to this kaleidoscopic succession of images but constructs true perceptual forms out of them. The "surface color" which we attribute to the "thing" as its property in contradistinction to the mere appearance of a "film color" or a "spatial color," represents just such a form. The "surface color" belongs invariably to the objects; it is not liable to variations produced by accidental changes of illumination; such variations would deprive it of all cognitive meaning. It is those very facts designated by us as fundamental phenomena of constancy of size, shape, and color, which preserve the cognitive value of perception.

Now the question arises which are the *means* that render this function of perception possible, and whether these means present some analogy to those of mathematical construction. The answer to this question might be indicated by the concept of "transformation" as employed in the modern psychology of perception. What is the significance of this concept with respect to the methodological foundation of geometry and with respect to psychology? As to the first question, we saw that it is just this concept which enables geometry to make the transition from the particular to the universal. Geometrical thought necessarily develops on the basis of concrete, particularized date of intuition. We mentioned the "crisis of intuition" which was

[22] Cf. Katz, 2nd edition, pp. 67 ff.; Gelb, *loc. cit.*, p. 599.

[23] Kaila, "Gegenstandsfarbe und Beleuchtung," in *Psychologische Forschung*, vol. III, 1923, p. 33.

brought about by the recent development of mathematical thought. If one considers this development, one must indeed admit that intuition has lost its predominant *logical* position and that it has sunk in importance as a *means* of geometrical *demonstration*. This, however, does not affect its significance as a point of departure. F. Klein, e.g., to whom we owe the generalization of geometrical concepts as outlined above, maintains that we still have to look upon "naive "geometrical intuition as the source of all fundamental geometrical concepts and axioms: "It is from intuition that we derive the data which, in appropriate idealization, are subject to logical treatment."[24] "To pursue geometrical reasoning in a purely logical way, without permanently keeping in front of my eyes the figure to which that reasoning applies, is, for me at least, impossible."[25] The problem is to conceive of this necessary *connection* with intuition in such a way that nonetheless the progress of the mathematical concept toward ultimate universality is left unimpeded. The solution of this problem, as offered by geometrical thought itself, shows us how this connection may be prevented from becoming restrictive. Taking our departure from a fact given in intuition, there are altogether different directions in which we may proceed and determine that fact accordingly, i.e., according to the group of transformations to which we refer. We enjoy complete freedom in the choice of these alternative groups. Different groups will yield different invariants and hence different geometrical properties. In familiar Euclidean geometry, the diverse conics, viz., the circle, the parabola, the ellipse, the hyperbola, are not only *intuitively* distinct, but also *conceptually* distinct. These distinctions disappear if, instead of choosing the "principal group" of Euclidean geometry, we choose the group of "affinitive or projective transformations." If, furthermore, higher point-transformations, the transformations by "reciprocal radii," the transformation with change of the spatial element are permitted, it appears that there is no limit to that progress toward universality. For *Analysis situs*, e.g., there is no such thing as what is usually meant by "identity of shape." A given shape is here regarded as the "same" in spite of all sorts of continuous distortions it may undergo. The question is no longer raised whether a given line is "straight" or "curved," whether a given length is equal to or the double of another length.[26] The nature of a given geometry is, then, defined by the *reference* to a determinate group and the way in which spatial forms are related within that type of geometry.

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[24] F. Klein, *Elementarmathematik vom hoeheren Standpunkt aus*, 3rd edition, Berlin, 1925, vol. II, p. 225.

[25] "Zur Nicht-Euklidischen Geometrie," *Mathematische Annalen*, vol. XXXVII, 1890; cf. *Gesammelte Mathematische Abhandlungen*, vol. I, p. 381.

[26] For details see Klein, *Elementarmathematik*, vol. II, Part II, pp. 74 ff.

The phenomena of perceptual constancy reveal a similar kind of reference in the domain of pure perception, which, although it exists, so to speak, "in *statu nascendi*" only, determines the structure of perception to a considerable extent. In perception, too, we do not confine ourselves to the particular, given *hic et nunc*, to be completely absorbed and, as it were, lost in it. We go beyond the particular and integrate it into a certain context. As the particular changes its position in the context, it changes its "aspect." We do not apprehend the particular as a mere "existence," that simple reality in which there corresponds a particular sensation to each particular stimulus. On the contrary, the apprehension of the particular *qua* "existence" involves apprehension of the possibilities of transformation which it contains within itself. The *perceived* phenomenal color differs from that "reduced" color-experience which corresponds to the retinal image. The former is conditioned and modified by the "perspective of illumination," in essentially the same way in which our visual perception of space is conditioned by the spatial perspective.[27] One might say that each particular perception assumes, with respect to the particular perspective involved, a definite index and, owing to the latter, a new *dimension*. Thus an achromatic color, e.g., may be seen as the same color through variation of the conditions of illumination; the latter does not effect the color as such, but only its "pronouncedness" (*Ausgepraegtheit*). The "same" grey or white color may appear in different degrees of pronouncedness.[28] In Hering's well-known experiment we experience the "shift" that occurs when a part of the field, being objectively darker than its environment, first appears as a spot and then as a shadow fallowing on the surface, and thus gives, while it is exposed to the same illumination as the white surroundings, first the impression of grey and then the impression of shadowy white. Here again the typical possibility of double orientation or reference is apparent. "There is a difference of *quality* between the sheet of paper and the spot, and a difference of *intensity* between the sheet of paper and the shadow: the sheet of paper is *white* and the spot is *grey*; the sheet of paper is *light* and the shadow is *dark*."[29] Helmholtz thought that according to the kind and intensity of the actual conditions of illumination or of what we believe these conditions to be, we apply different standards to our sensations of light and, correspondingly, change our judgments about external objects. What is the nature of such a standard?

If I am not mistaken it rests on the very same factor which is most ex-

[27] With regard to the concept of "perspective of illumination," cf. Buehler, *loc. cit.*, pp. 84 ff., and the experiments discussed by Katz, in *Aufbau der Farbwelt*, pp. 112 ff.

[28] Katz, *loc. cit.*, pp. 112 ff.

[29] Buehler, *Handbuch*, p. 117.

plicit and striking in the formation of geometrical concepts. The perceptual image as well involves that reference to certain possible groups of transformation. It changes when we refer it to a different group and determine the "invariants" of perception accordingly. In addition to Hering's shadow-experiment and photometer-experiment,[30] we may mention all those facts that are by Gestalt-Psychology described in terms of the category of "figure and ground."[31] All these phenomena are remarkably analogous to the above-mentioned different possibilities of "coordination" in Euclidean, affinitive, projective, etc., geometries. Thus Katz, in referring to certain observations, asserts that under the same objective conditions perception may shift from one mode of "apprehension" to another by distributing light and shadow in a different way. At one time we see shadows falling upon a light ground, at another time we see light falling upon a dark ground; and we are free to choose either mode of apprehending.[32] It is this free choice, and the perceptual structure which it determines, which represents what, on a higher level, we find in the formation of geometrical concepts, when such formation attains to a maximum of "spontaneity."

It goes without saying that this analogy between the formation of invariants in perception and in geometry ought not to make us overlook the thoroughgoing differences which are very important from the epistemological point of view. These differences may be characterized by an expression which Plato used to define the opposition of perception to thought. All perception is confined to the "more or less," the μᾶλλον τε καὶ ἧττον. Only approximative, not absolute determinations are attainable in perception. This characteristic is also exhibited by perceptual *constancy*. Its realization is never ideally complete, but always remains within certain limits. The fixation of these limits constitutes one of the most important tasks of psychological research.[33] Beyond these limits there is no further "transformation." The relative constancy of the color tone, for example, is destroyed when the color of illumination becomes too intense; correspondingly, in the degree as one's vision becomes more indirect, color constancy decreases to a considerable extent. In this connection Katz' laws concerning the extension of the field are very significant; they express that there is no constancy unless the conditions of illumination can be perceived in their totality.[34] There is no "total constancy of color," no "ideal in-

[30] *Grundzuege der Lehre vom Lichtsinn*, pp. 8, 15.

[31] Cf. E. Rubin, *Visuell wahrgenommene Figuren*, German edition, 1921.

[32] Katz, *Aufbau*, p. 202; cf. Buehler *loc. cit.*, pp. 81 ff.

[33] The investigations into the nature of perceptual objects, upon which Egon Brunswik relies for his *Grundlegung einer Psychologie vom Gegenstand her*, are for the most part concerned with just this fixation of limits; cf. Brunswik's *Wahrnehmung und Gegenstandswelt*, Leipzig and Vienna, 1934.

[34] Katz, *op. cit.* p. 50, pp. 343 ff.

variance," there is but a tendency in this direction. The constancy of size of visible objects also holds within limits only. We may again express this state of affairs in Platonic terms: the phenomenon *tends* toward the idea but never reaches it and necessarily falls short of it. Both "tending" and "falling short" are characteristic traits of perception. Only the mathematical concept renders a new orientation possible, viz., the orientation toward the "idea." Mathematical concepts are independent of any limits that might be imposed upon perception. The geometrical concept embraces and comprehends the totality and unlimited variety of modifications which a spatial figure undergoes when it is subjected to certain transformations. Once the group of transformations is specified, all the modifications that are possible with respect to this group can be determined by means of exact laws. Thus the transition from *esse* = *percipi* to *esse* = *concipi* is accomplished. This is the step which separates the "naive" idea of perception from the ideal of scientific knowledge. To perceive is to "evaluate," and evaluation cannot go beyond a certain "more or less"; it is necessarily vague and unprecise. The mathematical concept opposes to this lack of precision the postulate of exactness and accurate determination; it develops methods by which this postulate may be satisfied. On the other hand, the mathematical concepts are only the full actualization of an achievement that, in a rudimentary form, appears also in perception. Perception too involves a certain invariance and depends upon it for its inner constitution.

In order to elucidate these facts, let us consider Helmholtz' views once more. The problem with which we are confronted may be said to be almost the central problem of Helmholtz' psychological inquiries. What characterizes these inquiries and renders them *philosophically significant*, is the fact that Helmholtz discusses successively every possible aspect of the problem of perception, thus exhibiting his perfect mastery in every field concerned. Throughout his analyses, he starts as a physiologist and psychologist, and terminates as a mathematician. We have Helmholtz' own clear testimony concerning his development. In 1868, his attention became for the first time focused upon Riemann's inquiries into the foundations of geometry. In a letter to Schering, Helmholtz writes: "For the last two years, I have been dealing with the same problems in connection with my research in physiological optics, but I have not finished and published my work, because I had been hoping to be able to generalize on several points. Now, from the few hints you give me concerning the results of Riemann's inquiries, I see that the conclusions he has reached coincide exactly with mine.[35] The following question constitutes my starting-point: What must

[35] We may mention that in Riemann's fundamental work there is also a combination of psychological and mathematical points of view. In philosophy Riemann looked upon himself as a pupil of Herbart and was stimulated by Herbart's theory of the psychological formation of series.

be the nature of a multi-dimensional aggregate which is such as to permit everywhere continuous, monodromic, and free movements of solid bodies (i.e., bodies with constant relative size), like the movements of bodies in real space?"[36] This passage reveals a remarkable interconnection between various intellectual trends in Helmholtz' doctrine of space. Helmholtz tries to be an empiricist in geometry in order to be a geometrician in empirical psychology. On the problem of space, his empiricism exhibits a decidedly mathematical character. Far from explaining our intuition of space in sensationistic terms and deriving it from mere "sensation," he traces it, on the contrary, to a complicated tissue of "unconscious inferences" contrived by analogy to mathematical operations. According to Helmholtz, perceptual space originates from a kind of unconscious mathematics. As a variation upon the ancient theme *Cum Deus calculat, fit mundus*, one might paraphrase Helmholtz' doctrine thus· *Cum homo calculat, fit spatium*. Yet, *this* way of synthetizing mathematics and psychology is questionable from two points of view. From the psychological viewpoint it is open to the objection that it does not do justice to the phenomena as they are disclosed in simple observation. By being transferred into the unconscious, the problem becomes inaccessible to phenomenological analysis. Instead of an analysis of observable facts, we are left with an hypothesis which is, at best, amenable to indirect verification. It is on this point that criticisms were raised, especially by Hering. Hering did not grow tired of pointing out the flaws in Helmholtz' exposition of the perceptual facts. He emphatically insists that it is not in virtue of our *knowledge* of differences in external conditions, but in virtue of an essential difference in the very act of *vision*, that we are able to phenomenally distinguish between film colors and the colors of objects.[37] On the other hand, Helmholtz' theory does not do full justice to the mathematical facts either. Just as Hering had to correct it from the standpoint of psychology, thus Poincaré had to correct it from the standpoint of geometry. The axioms of geometry cannot be interpreted as empirical statements; such an interpretation would fail to grasp their proper meaning and logical status. The axioms refer to determinations that are never given or realized in experience. Thus experience can neither validate nor invalidate them. The axioms cannot be derived from physical reality, but must be constructed in full independence of such reality; they refer to *possibilities* only. Experience cannot determine these constructions. It may, however, to some extent define the direction they take, in so far as it represents the *occasion* for the purely logical construction of such systems of axioms as correspond and are applicable to certain empirical

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[36] Letter to Schering, April 21, 1868, published by L. Koenigsberger, H. Helmholtz, vol. II, Braunschweig, 1903, p. 138.

[37] Hering, *Grundzuege*, p. 4, p. 8.

situations. Axioms may thus refer *to*, but they do not derive *from* experience. If we reflect upon both types of objection, the psychological and the mathematical, we realize *that* and *why* we must look for the synthesis of mathematics and psychology, which Helmholtz tried to achieve both as a philosopher and as a mathematician, in some other direction. The direct road which he attempted to travel cannot lead to the goal, for there cannot possibly obtain an immediate correspondence between psychological and mathematical "facts." But we might approach our goal in an indirect way, with the help of a mediating principle of a higher order. Instead of following in the footsteps of geometrical empiricism, such as to search for the "facts which lie at the basis of geometry," we may raise the question whether there are any concepts and principles that are, although in different ways and different degrees of distinctness necessary conditions for both the constitution of the perceptual world and the construction of the universe of geometrical thought. It seems to me that the concept of group and the concept of invariance are such principles. Perhaps we can, by their instrumentality, bring certain mathematical and psychological problems under a common denominator—although in quite a different way than Helmholtz attempted to achieve such a synthesis.

The very phenomenon of perceptual *constancy* shows clearly that the process of perception is not a process of mere *reproduction*. The theory of *tabula rasa* is just as inadequate to account for "reflection" as it is to account for pure "sensation." We cannot compare perception to the reception of light by a photographic plate[38] and the development of an image that is exclusively determined by the light falling on the plate. Only in rare, exceptional cases, under the artificial conditions of "reduction," does this ever happen. There seems to be no stage, however "primitive," of perception, at which perception constantly reacts to the "same" stimulus by producing the "same" sensation. The experiments performed by W. Koehler, Burkamp, Katz, and others on animals have revealed the existence of constancy of size or color even within animal perception.[39] This shows that wherever there is an opposition and separation between an "ego" and the "world," between "subject" and "object," perception is something altogether different from mere reflection of the "external" by the "internal." Perception is not a process of reflection or reproduction at all.

[38] In modern psychology of perception this kind of comparison has been continued and developed in Russell's *Analysis of Mind*, London, 1921, esp. pp. 99 ff. I cannot here go into Russell's views, but refer to my detailed criticism in *Jahrbuecher der Philosophie*, edited by Frischeisen-Koehler, vol. III, 1927, pp. 52 ff.

[39] W. Koehler, *Abhandlungen der Berliner Akademie der Wissenschaften, Mathematisch-Physikalische Klasse*, III, 1915; Katz and Revesz, *Zeitschrift fuer angewandte Psychologie*, vol. XVIII, 1921; Burkamp, *Zeitschrift fuer Sinnesphysiologie*, vol. LV, 1923. Cf. the final summary in Katz, *Aufbau der Farbwelt*, pp. 418 ff.

It is a process of objectification, the characteristic nature and tendency of which finds expression in the formation of invariants. It is within this process that the distinction between "reality" and "appearance" emerges. We construct the "true" color out of the appearances due to the conditions, of illumination, we construct the "true" size of the object out of the apparent size of the retinal image.

This rudimentary tendency toward "objectification" reappears in conceptual, in particular mathematical, thought, where it is developed far beyond its primitive stage. When we determine the size of an object by *measurement*, it is owing to such "objectification" that we succeed in transcending the accidental limits of our bodily organization. It enables that elimination of "anthropomorphic elements" which is, according to Planck, the proper task of scientific natural knowledge. To geometrical invariants have to be added physical and chemical constants. It is in these terms that we formulate the "existence" of physical objects, the objective properties of things. Also Helmholtz concerned himself with this problem of the relation between different stages of objectification, in his essay *On the origin and meaning of geometrical axioms*. In this essay he endeavors to find out how the various measuring processes that enter into perception are related to geometrical measurement. "When we perform measurements, we do but employ the best and most reliable means we know of in order to determine what we habitually determine by forming an estimate by sight, touch, or steps. In these habitual measurements it is our own body with its organs which is the measuring instrument we carry around with us in space. Now it is our hands, then our legs, which serve as a compass, or our eyes, turning in all directions, are our theodolite for measuring arcs and angles in the visual field."[40] Geometrical concepts presuppose such measurements by means of our body and sense-organs; but they render them exact and objectively valid. Hering, too, saw himself confronted with the problem of objectification in the process of perception. He gives not only a psychological description and physiological explanation of the phenomena of constancy, but moreover undertakes to determine their teleological significance, the function they perform in our knowledge of the external world. "What matters in the visual process is not the perception of the radiations as such, but the perception of the external objects, mediated by these radiations; it is not the function of the eye to inform us about the intensity or quality of the light that is reflected from external objects, but to inform us about those very objects." The eye could not fulfill this function unless it possessed the capacity of discriminating within the visual experience between "illumination" and "that which is illuminated."[41] Hering speaks

[40] *Popular Lectures on Scientific Subjects*, London, 1908, II, p. 56.

[41] *Grundzuege der Lehre vom Lichtsinn*, pp. 13 ff.

here the language of the scientist, i.e., of realism. He assumes the empirical reality of the objects about which our senses have to inform us. But a critical analysis of knowledge must go farther. Such an analysis reveals that the "possibility of the object" depends upon the formation of certain invariants in the flux of sense-impressions, no matter whether these be invariants of perception or of geometrical thought, or of physical theory. The *positing* of something endowed with objective existence and nature depends on the formation of constants of the kinds mentioned. It is, then, inadequate to describe perception as the mere mirroring in consciousness of the objective conditions of things. The truth is that the *search for constancy*, the tendency toward certain invariants, constitutes a characteristic feature and immanent function of perception. This function is as much a condition of perception of objective existence as it is a condition of objective knowledge.

IV

The group-theoretical interpretation of the fundaments of geometry is, from the standpoint of pure logic, of great importance, since it enables us to state the problem of the "universality" of mathematical concepts in simple and precise form and thus to disentangle it from the difficulties and ambiguities with which it is beset in its usual formulation. Since the times of the great controversies about the status of universals in the Middle Ages, logic and psychology have always been troubled with these ambiguities. Berkeley tried to cut the Gordian knot. He wanted to solve the problem by showing that it was an artificial pseudo-problem. If geometry were to deal with "abstract ideas," it could yield no truth and no scientific knowledge of objective validity. An "abstract idea" is devoid of any real content to be known. It is an *ens imaginarium*, a mere fiction. A "universal triangle" would have to be represented as being at once right-angled, acute-angled, and obtuse-angled, and as having all at once an indefinite number of sides of different length, of possible positions in space, etc. Upon reflection on the nature of such a representation and realization of its psychological conditions, the inner contradiction involved in these conditions must spring into our eyes. The "abstract idea" thus appears once and for all as a "squared circle"; it is a verbal construct devoid of concrete reference and incapable of psychological realization.

The fallacy of this argument lies in an obvious *petitio principii*. The principle which Berkeley takes for granted without proof is the principle of sensationistic psychology, according to which there is but one mode of psychological realization, viz., immediate "impressions" or representative images derived from those impressions as their "copies." If, by virtue of a psychological *axiom*, the idea is defined as a "copy of sense-impressions," the notion of a *general idea* does, of course, involve a palpable absurdity. Yet, we have but to abandon this axiom, and the problem to be solved, as well as its solution, will assume an altogether different form. Kant replaces the sensationistic deduction of the concept by a "transcendental" deduction, showing that the concept cannot be represented in the form of an *image*, but only in the form of a rule. The rule possesses that generality to which the image cannot possibly attain. This is the conclusion which Kant reaches in the chapter on the Schematism of the Pure Concepts of the Understanding, and by means of which he tries to avoid Berkeley's *aporiae*.[42] Concepts are psychologically actualized by "schemata," not by images. In fact, "no image could ever be adequate to the concept of a triangle in general. It would never attain that universality of the concept which renders it valid for all triangles, whether right-angled, obtuse-angled, or acute-angled; it would always be limited to a part only of this sphere. The schema of the triangle can exist nowhere but in thought and signifies a rule of synthesis of the imagination, in respect to its figures in space." The same conclusion holds true not only of the pure concepts of geometry but also of our empirical concepts. If we want to look upon the latter as genuine "concepts," i.e., as endowed with objective validity, we cannot put them together out of mere impressions and think of them as aggregates of impressions. It is not possible to realize the thought of a perceptual *object*—the intended "object" of perception—in perceptual consciousness by a mere image; it cannot be represented except by a *rule*: "The concept 'dog,' for instance, signifies a rule according to which my imagination can delineate the figure of a four-footed animal in a general manner, without limitation to any single determinate figure such as experience, or any possible image that I can represent *in concreto*, actually presents."[43]

Our foregoing reflections on the concept of *group* permit us to define more precisely what is involved in, and meant by, that "rule" which renders both geometrical and perceptual concepts universal. The rule may, in simple and exact terms, be defined as that *group of transformations* with regard to which the variation of the particular image is considered. We have seen above that this conception operates as the constitutive principle in the construction of the universe of mathematical concepts. If, for the definition of the triangle, the square, the ellipse, the parabola, etc., the geometrician had to depend upon constructing thse figures from varying images of triangles, squares, etc., and upon having all the elements of these images blend with each other, he would, indeed, be confronted with a problem that is

[42] In my *Erkenntnisproblem* (3rd edition, vol. II, pp. 713 ff.) I tried to show that the chapter on the Schematism is closely related, both logically and historically, to Berkeley's problem of the nature of concepts.

[43] Kant, *Kritik der reinen Vernunft*, 2nd edition, p. 180.

logically impossible and psychologically insoluble. But it is quite a different matter to start from the intuition of a given concrete figure and at the same time to conceive in the latter the totality of possible transformations to which it may be subjected according to certain laws of transformation. In Euclid's classical geometry these laws were subject to some limitation in so far as they, though being conceived with perfect generality, had to satisfy the additional postulate that every phase of the process of transformation must be open to intuitive inspection. Within Euclidean geometry, a "triangle" is conceived of as a pure geometrical "essence," and this essence is regarded as invariant with respect to that "principal group" of spatial transformations to which Euclidean geometry refers, viz., displacements, transformations by similarity. But it must always be possible to exhibit any particular figure, chosen from this infinite class, as a concrete and intuitively representable object. Greek mathematics could not dispense with this requirement which is rooted in a fundamental principle of Greek philosophy, the principle of the correlatedness of "logos" and "eidos." It is, however, characteristic of the modern development of mathematics, that this bond between "logos" and "eidos," which was indissoluble for Greek thought, has been loosened more and more, to be, in the end, completely broken. Since Descartes' discovery of analytic geometry, geometrical concepts have assumed an algebraic, and hence analytic, character.[44] Since the beginning of the nineteenth century, a strong reaction against this "arithmetization" of geometry has set in. The founders of projective geometry offer strong resistance to the dissolution of space into number; they want to maintain the conceptual generality of the geometrical without sacrificing its proper meaning and autonomy. Poncelet was the first one to give a precise statement of this requirement. His principle of continuity, which is the basis of his method of treating geometrical problems, amounts to a methodological postulate rather than a constitutive axiom. His procedure is to start from the consideration of certain figures and to vary these figures according to certain rules while preserving certain fundamental relations. The second step is to embrace the *totality* of these variations with a single glance and to subject this totality, as a geometrical construction, to investigation.[45] In order to lay down and fulfill this postulate, Poncelet had to break with the traditional approach to geometrical problems. He had to emancipate geometrical thought from all connection with "elements" that could be given in intuition, and to consider the *rela-*

[44] With regard to this change, I refer especially to the systematic and historic exposition in the works of Pierre Boutroux, *L'Idéal Scientifique des Mathématiciens*, Paris, 1920, and *Les Principes de l'Analyse Mathématique*, 2 vols., Paris, 1914 ff.

[45] Cf. Poncelet, *Traité des Propriétés Projectives des Figures*, Introduction, Paris, 1822.

tions between these elements as the proper and only subject-matter of geometrical knowledge. Thus the construction of geometrical concepts acquired a new kind of freedom, as compared with the way geometry was handled by the ancients. In Poncelet's work, this freedom manifests itself especially in the introduction of the imaginary and the use made of it in the construction of projective geometry. This process has come to its logical conclusion and systematic completion in the development of modern group-theory. Geometrical *figures* are no longer regarded as fundamental, as date of perception or immediate intuition. The "nature" or "essence" of a figure is defined in terms of the *operations* which may be said to *generate* the figure. The operations in question are, in turn, subject to certain group conditions. Lie and Klein have shown that the characteristic properties of an aggregate are determined only by the group and not by the elements out of which the aggregate is constructed.[46] The figures that belong to a given group constitute a unity, no matter whether and how they be representable in an intuitive way. For instance, it is characteristic of the "dualistic transformations," which play an important role in projective geometry, that they allow figures of altogether different kinds to be transformed into one another. A theorem about points and lines is not modified, if, according to the principle of duality, the words "point" and "line" are mutually interchanged. For modern geometry, two figures related by duality are no longer different but identical. A further novelty is represented by the notion of *imaginary transformations*; the reason for their being introduced does not concern the group of projective and dualistic transformations, but algebraic operations.[47]

It is hence obvious that mathematical theories have developed in spite of the limits within which a certain psychological theory of the concept tried to confine them. Mathematical theory ascended higher and higher in order to look farther and farther. Again and again it ventured the Icarian flight which carried it into the realm of mere "abstraction" beyond whatever may given and represented in intuition. It must be admitted that Berkeley foresaw this development. His admonitions against it are understandable if one considers his basic psychological and epistemological convictions. What he violently attacked in "the Analyst" was the new analytic spirit which he saw arising in Leibniz' infinitesimal calculus and Newton's method of fluxions. Did Berkeley's doctrine prove adequate even within its proper domain, viz., the psychology of perception? Is his doctrine acceptable, if not for the characterization of mathematical concepts, for the description of

[46] Cf. L. Maurer and H. Burckhardt, "Kontinuierliche Transformationsgruppen," *Enzyklopaedie der Mathematik*, II A 6; vol. II, Part I, pp. 401 ff.

[47] Cf. Klein, "Erlanger Programm," *Gesammelte Mathematische Abhandlungen*, vol. I, pp. 465 ff.

the phenomena of pure "perception"? For a long time it seemed as though this question had to be answered in the affirmative. During the first half of the nineteenth century, the psychology of perception was almost completely dominated by the fundamental ideas of Berkeley and Hume. Empirical psychologists hardly ever expressed any doubt as to the adequacy of the concepts of "sensation" and "associative connection" for the theoretical formulation and solution of all problems that concern the universe of immediate perception. The situation was radically changed when Ehrenfels introduced the concept of "form-qualities" (*Gestaltqualitaeten*) in his well-known essay. He illustrates this concept especially by melodies and the similarity of certain optical figures. "It is characteristic of phenomenal forms (*phaenomenale Gestalten*) that their specific properties remain unchanged when the absolute data upon which they rest undergo certain modifications. Thus a melody is not substantially altered when all of its notes are subjected to the same relative displacement; an optical spatial figure remains approximately the same when it is presented in a different or on a different scale, but in the same proportions." It is in these terms that the phenomenon dealt with by Ehrenfels was later formulated by W. Koehler.[48] However, this phenomenon is related to a much more general problem, a problem of abstract mathematics. Indeed, what else is that "identity" of the perceptual form but what, in a much higher degree of precision, we found to subsist in the domain of geometrical concepts? What we find in both cases are invariances with respect to variations undergone by the primitive elements out of which a form is constructed. The peculiar kind of "identity" that is attributed to apparently altogether heterogeneous figures in virtue of their being transformable into one another by means of certain operations defining a group, is thus seen to exist also in the domain of perception. This identity permits us not only to single out elements but also to grasp "structures" in perception. To the mathematical concept of "transformability" there corresponds, in the domain of perception, the concept of "transposability." The *theory* of the latter concept has been worked out step by step and its development has gone through various stages.[49] Whatever were the terms in which this theory was formulated, it appeared again and again that even for the very *description* of the novel phenomenon with which the theory of perception has confronted us it is necessary to abandon the pattern of sensation and association, laid down by the classics of sensationalism. Gestalt psychology made the attempt to

[48] W. Koehler, *Die physischen Gestalten in Ruhe und im stationaeren Zustand*, 1920, p. 37.

[49] This development has been surveyed historically and systematically by E. Brunswik in his essay "Prinzipienfragen der Gestalttheorie," *Festschrift zu Karl Buehler's 50. Geburtstag*, Jena, 1929, pp. 78 ff.

give such a description on an altogether new basis. By the acceptance of "form" as a primitive concept, psychological theory has freed it from the character of *contingency* which it possessed for its first founders. The interpretation of perception as a mere mosaic of sensations, a "bundle" of simple sense-impressions has proved untenable. It has been laid down as a general *principle* of psychological research that the soul and the psychophysical organism of stimuli-reception are not "receptors" like mirrors or cameras, i.e., receive separate "stimuli" and combine them into comprehensive wholes that have the character of mere aggregates. If perception is to be compared to an apparatus at all, the latter must be such as to be capable of "grasping intrinsic necessities."[50] Such intrinsic necessities are encountered everywhere. It is only with reference to such "intrinsic necessity" that the "transformation" to which we subject a given form is well defined, inasmuch as the transformation is not arbitrary and executed at random but proceeds in accordance with some rule that can be formulated in general terms. In the domain of mathematics this state of affairs manifests itself in the impossibility of searching for invariant properties of a figure except with reference to a group. As long as there existed but *one* form of geometry, i.e., as long as Euclidean geometry was considered as *the* geometry *κατ' ἐξοχήν* this fact was somehow concealed. It was possible to assume *implicitly* the principal group of spatial transformations that lies at the basis of Euclidean geometry. With the advent of non-Euclidean geometries, however, it became indispensable to have a complete and systematic survey of the different "geometries," i.e., the different theories of invariancy that result from the choice of certain groups of transformation. This is the task which F. Klein set to himself and which he brought to a certain logical fulfillment in his *Vergleichende Untersuchungen ueber neuere geometrische Forschungen*.

Thus, however, we seem again to be led to a point where the analogy between the invariants of perception and those of geometry disappears. That form of logical *systematization* which is both possible and necessary in the domain of geometrical thought is once and for all inaccessible to perception. Here we have to take the phenomenal facts as they present themselves in experience; we cannot go beyond the simple ascertainment of these facts. It is but empirical observation that can tell us in which domains of sense-perception there exist phenomena of constancy and how far their influence extends. Here no *a priori* judgment is possible. However, it is important not to confuse the empirical discovery of facts with their empiricistic explanation. As far as I can see, the latter has been increasingly abandoned by modern psychological theories. Katz, in his first investiga-

[50] Cf. Max Wertheimer, "Untersuchungen zur Lehre von der Gestalt II," *Psychologische Forschung*, vol. IV, 1924, p. 349.

tions into the "phenomenal aspect of color," ascribed to the "experience of the individual" some influence on the production of the phenomenon of constancy, but later on he minimized the importance of this factor.[51] The fact, moreover, that the phenomena in question extend far down into the animal realm and seem to appear at very primitive stages of evolution, does not favor their explanation in terms of the experience of the individual. "What we encounter in our own perceptions, as for instance constancy of visual appearances through variations of illumination, or constancy of size of seen objects through variations of distance." writes Buehler, "is, according to all that we know about it, no condition restricted to human experience and only acquired by man, but a common property of at least the whole realm of vertebrates."[52] This seems to suggest a biological deduction and explanation, based on the theory of "Mneme" which R. Semon had introduced into biology. Even Hering was not far from such an interpretation; he explicitly pointed to memory as a general function which we have to take account of in all our explanations of biological phenomena.[53] Why not interpret the phenomena of constancy as products of experience as far as the experience of the species is concerned? Could not they be conceived as accumulations and accretions of a manifold of particular impressions, inscribed in memory in the form of certain "engrammata," to use Semon's expression, and transmitted to the descendants? To be sure, this would not be an *empirical* explanation in the strict sense of the term. Obviously the function ascribed to memory is not an ascertained fact but a hypothetical inference. The latter is all the more objectionable as it involves us in all the difficulties that beset the theory of "hereditary transmission of acquired characters," one of the most difficult and most controversial problems of modern biology. The more one studies the phenomenon of perceptual constancy, the more its explanation by "experience" proves unsatisfactory, in so far, at least, as by experience is meant a juxtaposition of particular items, an accumulation of mere accidents. In order to be able to develop at all, Gestalt psychology had to abandon this conception of experience. It replaced it by the concept of original "Gestalt-dispositions," tendencies toward something like "good shape" and concrete "laws of organization."[54] We have instances of such "good shapes," to which individual impressions are oriented, in those sense data which are grasped and retained in perception as the "true size" or as the "true color"

[51] Cf. the preface to the *second edition* of Katz' work and his discussion with Gelb, pp. 453 ff.

[52] Buehler, *Die Krise der Psychologie*, pp. 81 f.

[53] Cf. Hering's work: *Ueber das Gedaechtnis als allgemeine Funktion der organischen Materie*, Vienna, 1876.

[54] Cf., e.g., Wertheimer, *Psychologische Forschung*, vol. I, p. 53.

of an object. By their reference to such "good" points, the particular impressions receive a new kind of determination. They lose, so to speak, their atomicity, their uniqueness as mere particular items; they unite into groups and totals. As far as the perception of colors is concerned, Helmholtz stressed our capacity of correcting colors that are presented in unusual illumination; we "see" these colors as though they appeared under normal conditions of illumination. Impressions received by peripheral parts of the retina are translated into those that would result from direct perception of the object by means of the center of the retina. It is upon such translations and transformations that the existence of our "objective" intuitions depends. Reference to *typical* configurations proves to be one of the essential conditions of the process of spatial objectification. As William James puts it in his felicitous manner: "In our dealing with objects, we always do pick out one of the visual images they yield, to constitute the real form or size."[55]

James expressed the same idea in speaking of "the choice of the visual reality." In perception we are, according to him, constantly making selections from among the vast manifold of utterly heterogeneous optical impressions that strike our sense-organs. These impressions differ in their value for the construction of our representation of the objective world. For this construction we give preference to a certain *class* of phenomena which hence assume a privileged position. These phenomena—for example, the spatial forms that appear in a vision by means of the central area of the retina—receive a typical value. They become centers of reference and these centers define a kind of norm, a standard of measurement which determines the objective meaning of every impression. It makes a difference whether we experience a certain phenomenon of light in this or that "mode of appearance." It is not the same thing to see a light falling on an object as luminosity as it is to see this light as color; nor is it the same thing to perceive some darkness on the object as a shadow as it is to perceive it as a spot. When we pass from one mode of perception to the other, we experience that characteristic shift which Hering describes in his well-known "shadow-experiment."[56] One and the same phenomenon appears differently as far as its objective significance is concerned. The shadow is taken

[55] *The Principles of Psychology*, 1901, vol. II, p. 238.

[56] "If I hang up . . . a scrap of paper by a silk thread so that by means of a fittingly placed small incandescent lamp it throws a faint shadow on my paper, I see the shadow as something dark which happens to be on the white paper. But when I draw a broad black line around that shadow such as to cover the penumbra completely, I see a grey spot inside of the black contour, just as though the white paper were here colored gray by drawing-ink, or as though a grey paper with a black margin were glued on the white paper."

as a fickle, transient phenomenon that depends upon external circumstances, whereas the spot is considered as stable and somehow connected with the "substance" of the seen object. Without discrimination between the accidental and the substantial, the transitory and the permanent, there would be no constitution of an objective reality.

This process, unceasingly operative in perception and, so to speak, expressing the inner dynamics of the latter, seems to have come to final perfection, when we go beyond perception to enter into the domain of pure thought. For the logical advantage and peculiar privilege of the pure concept seems to consist in the replacement of fluctuating perception by something precise and exactly determined. The pure concept does not lose itself in the flux of appearances; it tends from "becoming" toward "being," from dynamics toward statics. In this achievement philosophers have ever seen the genuine meaning and value of geometry. When Plato regards geometry as the prerequisite to philosophical knowledge, it is because geometry alone renders accessible the realm of things eternal; *τοῦ γὰρ ἀεὶ ὄντος ἡ γεωμετρικὴ γνῶσίς ἐστιν.* Can there be degrees or levels of objective knowledge in this realm of eternal being, or does not rather knowledge attain here an absolute maximum? Ancient geometry cannot but answer in the affirmative to this question. For ancient geometry, in the classical form it received from Euclid, there was such a maximum, a *non plus ultra.* But modern group theory thinking has brought about a remarkable change in this matter. Group theory is far from challenging the truth of Euclidean metrical geometry, but it does challenge its claim to definitiveness. Each geometry is considered as a theory of invariants of a certain group; the groups themselves may be classified in the order of increasing generality. The "principal group" of transformations which underlies Euclidean geometry permits us to establish a number of properties that are invariant with respect to the transformations in question. But when we pass from this "principal group" to another, by including, for example, affinitive and projective transformations, all that we had established thus far and which, from the point of view of Euclidean geometry, looked like a definitive result and a consolidated achievement, becomes fluctuating again. With every extension of the principal group, some of the properties that we had taken for invariant are lost. We come to other properties that may be hierarchically arranged. Many differences that are considered as essential within ordinary metrical geometry, may now prove "accidental." With reference to the new group-principle they appear as "unessential" modifications. Thus, as mentioned already, when we pass from ordinary to affinitive geometry, the difference between circle and ellipse vanishes, both being taken as *one* figure. When we pass to projective geometry, we meet still with a further restriction upon what may be considered as an "essential" geometrical property. Now even the difference between the circle and all other conics must be abandoned. In projective geometry there is but one single conic; for any two conics are transformable into a circle and hence also into each other. From this point of view, the difference between ellipse, parabola, and hyperbola is no longer absolute; it concerns but the accidental position with respect to some line considered as "infinite." In the "geometry of reciprocal radii," for instance, the concepts of a line or a plane, which are fundamental for Euclidean geometry, have no more independent meaning; the line is subordinated to the circle, the plane to the sphere as special cases.[57] "The progressive separation of affinitive and projective geometry from metrical geometry," thus Klein comments upon this procedure, "may be compared to the procedure of the chemist who, by applying more and more powerful decomposers, isolates more and more valuable elements from a substance; our decomposers are first the affinitive, then the projective transformation." What is separated out by the latter kind of transformations is more "valuable" in so far as it proves invariant with respect to a wider group of possible changes. In affinitive and projective geometries, parallel and central projection is superadded to the principal group of transformations admitted in Euclidean geometry. "Analysis situs" leads us still farther in this direction. Considered from the modern point of view, "analysis situs" is the most general kind of geometry, the theory of purely topological relations, entirely independent of metrical relations. In Klein's phrase, it "results, so to speak, from the most powerful corrosion"; it considers the "totality of properties that are invariant with respect to all possible one-to-one continuous transformations." From the point of view of modern geometrical systematization, geometrical judgments, however "true" in themselves, are nevertheless not all of them equally "essential" and necessary. Modern geometry endeavors to attain progressively to more and more fundamental *strata* of spatial determination. The depth of these strata depends upon the comprehensiveness of the concept of group; it is proportional to the strictness of the conditions that must be satisfied by the invariance that is a universal postulate with respect to geometrical entities. Thus the objective truth and structure of space cannot be apprehended at a single glance, but have to be *progressively* discovered and established. If geometrical thought is to achieve this discovery, the conceptual means that it employs must become more and more universal.

There is no direct analogy between these achievements of mathematical thought and those of perception. There is no direct comparison between them possible, since no common measure applies to them. Helmholtz

[57] As to details I refer to F. Klein, *Elementarmathematik vom hoeheren Standpunkt aus*, vol. II, pp. 103 ff., 140 ff.

made the attempt to find such a common measure. For this purpose, he intellectualizes, as it were, perception by interpreting it in terms of "unconscious inference." But Helmholtz' attempt must be judged unsuccessful in the face of the data of experience. Nevertheless, we must not conclude that no *mediation* at all can obtain between these two levels. In spite of their specific differences they belong to the same *genus*, in so far as they share the *function* of objective knowledge. It is this common function whence their character derives. Without the "reference of ideas to an object," there is no perception. And even within perception we can discriminate different levels of construction. The intentional reference to an object is not, to the extent to which it is realizable at all in perception, fulfilled all at once, but gradually only. According to their positions and meaning within this series, different perceptions possess more or less "depth."[58] Different perceptions refer not only to the object in general, but, according to the degree of generality of the invariants that are seized upon under varying conditions of observation, they penetrate, so to speak, to objective strata of different depths. In this sense, the system of fundamental concepts with which Euclidean geometry presents us, has, as it were, an upward and a downward reference. If we proceed in the upward direction we come to the all-comprehensive geometrical systematization achieved by group theory; if we proceed in the downward direction, we encounter those "schemata" that are present already in perception and immediate intuition.

As to the historical aspect of our problem, the foregoing reflections lead us back to a question which we already touched upon. Is there any logical connection between the subject of our discussion and the question discussed by Kant in the chapter on the Schematism, in his *Critique of Pure Reason?* To be sure, the two problems cannot really be identified with each other, since they belong, methodologically speaking, to different dimensions. Kant's theory has a strictly "transcendental" orientation, and this remains true even when it concerns itself with psychological problems. For Kant, the schemata belong to the "transcendental doctrine of judgment"; in discussing them, he anticipates problems that find their systematic discussion and clarification only in the *Critique of Judgment.* The fact, however, that there is nonetheless a point of contact between Kant and modern psychology has been noted and commented upon by investigators interested in the philosophical foundations of a theory of perceptual constancy. As Buehler writes in his *Sprachtheorie*, "the concept of factors of constancy in the face of variations of both external and internal conditions of perception is the realization, in modern form, of that which in principle . . . was known to

[58] As to the concept of "perceptual depth" (*Wahrnehmungstiefe*: Buehler) and the "criteria of perceptual depth," cf. Brunswik, *loc. cit.*, pp. 48 ff., 101 ff.

Kant, the analyst, and which he stated in terms of mediating, ordering schemata."[59] What, then, is this relation, and which is its systematic foundation? Kant called the schematism "an art concealed in the depths of the human soul, whose real modes of activity nature will hardly ever allow us to discover, and to have open to our gaze." Has modern psychology in any way advanced toward a disclosure of this "concealed art," and in which direction is its contribution toward such a disclosure to be sought? We may venture to answer this question, reminding ourselves of Kant's characterization of the schemata as "monogrammata of pure imagination." "The image is a product of the empirical faculty of productive imagination, the schema of sensible concepts. . .is a product and, as it were, a monogram, of pure *a priori* imagaination, through which, and in accordance with which, images themselves first become possible. These images can be connected with the concept only by means of the schema to which they belong. In themselves they are never completely at one with the concept."[60] The schemata are *mono*grammata because they express an original function of unification. The "images" which we receive from objects, the "impressions" which sensationalism tried to reduce perception to, exhibit no such unity. Each and every one of these images possesses a particularity of its own· they are and remain discrete as far as their contents are concerned. But the analysis of perception discloses a formal factor which supersedes this particularity and disparity. Perception unifies and, as it were, concentrates the manifold of particular images with which we are supplied at every moment. Perception fits this stream of images into definite channels. It cannot be reduced to a mere manifold of impressions, the "polygrammata" of sensibility, in any more satisfactory manner than to a mere reproductive function in terms of "engrammata" of memory. Beyond these "polygrammata" and "engrammata" there appears a specific function of perception· the "monogram of imagination." Each invariant of perception is in fact such a "monogram," a schema toward which the particular sense-experiences are orientated and with reference to which they are interpreted.

Thus we are provided with an answer to a further question which has very often presented difficulties to historians of philosophy and psychology. In a well-known passage Kant writes· "Psychologists have hitherto failed to realize that imagination is a necessary ingredient of perception itself. This is due partly to the fact that that faculty has been limited to reproduction, partly to the belief that the senses not only supply impressions but also combine them so as to generate images of objects. For that purpose something more than the mere receptivity of impressions is undoubtedly

[59] Buehler, *Sprachtheorie*, Jena, 1934. p. 252.

[60] *Kritik der reinen Vernunft*, 2nd edition, pp. 180 f.

required, namely, a function for the synthesis of them."[61] It has been said by historians of psychology that Kant has here been led into a "strange historical error."[62] Indeed, does not the whole history of psychology show that just the opposite is the case, viz., that the role of "imagination"has never been overlooked nor underrated? Attention is drawn to this role as early as in the first systematic foundations of psychology. In fact, in Aristotle's περὶ ψυχῆς the concurrence of αἴσθησις, μνήμη, and φαντασία in the construction of the perceptual world is maintained with full precision. This concurrence is most emphatically stressed by modern rediscoverers and renewers of Aristotelian psychology: Campanella, Giordano Bruno, and Vives developed theories of imagination of their own. In the eighteenth century, Tetens, whose psychological views bear a remarkable similarity to Kant's, has pointed again and again to the significance of "Dichtungsvermoegen."[63] What, then, is Kant's discovery? Which is that factor which, according to Kant, "psychologists have hitherto failed to realize"?

To answer this question we must pay attention to the point emphasized by Kant himself in the quoted passage. What is important for Kant is not that the imagination intervenes in some way or other in the production of perceptual images, but the fact that images of *objects* are formed by the imagination and can be formed only in this manner. The emphasis is not on the problem of psychological genesis but on that of objective validity. In this respect, Kant treads a new path, breaking with the whole tradition of psychological empiricism. In Hume's theory, imagination occupies a central position. It is imagination on which rests our belief in the regularity of Nature, the connection between cause and effect, the continued existence of things beyond the moment of present actual perception. Hume never questioned this "belief" nor its paramount importance. He is a skeptic only in so far as he denies the objective *validity* of such a belief. To Hume, the imagination is a source, not of knowledge, but of error. He sees the effects of imagination, but the latter is and remains to him altogether irrational. "I cannot conceive," he writes in the *Treatise of Human Nature*,[64] "how such trivial qualities of the fancy, conducted by false suppositions, can ever lead to any solid and rational system. . .'Tis a gross illusion to suppose that our resembling perceptions are numerically the same, and 'tis this illusion, which leads us into the opinion that these preceptions

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[61] *Kritik der reinen Vernunft*, 1st edition, p. 120.

[62] Cf. Max Dessoir, *Abriss einer Geschichte der Psychologie*, Heidelberg, 1911, p. 151.

[63] *Philosophische Versuche ueber die menschliche Natur*, 1777. For further details about Tetens' concept of "Dichtungsvermoegen," cf. my work *Das Erkenntnisproblem*, 3rd edition, vol. II, pp. 567 f.

[64] Book I, Part IV, §2; edited by Selby-Bigge, pp. 217 f.

are uninterrupted, and are still existent, even when they are not present to the senses. This is the case with our popular system. As to our philosophical one, 'tis liable to the same difficulties. . .What then can we look for from this confusion of groundless and extraordinary opinions but error and falsehood? At this point Kant's solution of Hume's problem sets in. It is not only human understanding but also "imagination" which Kant attempts to rehabilitate from Hume's doubt; he wants to show that imagination is not destructive but constructive, that it is "productive imagination." For this purpose Kant establishes the theory of "schemata of imagination," showing that imagination, far from falsifying the images of objects, is, on the contrary, indispensable for objective determinations to be known as such. For Hume, imagination can have but a negative significance; for imagination leads us away from immediate truth given and contained in "simple perceptions." For Kant, truth does not lie in these simple perceptions, but in the system, in the "context" of experience in accordance with general laws. For Kant, imagination is the first and necessary step towards generality; the intuitive schemata of imagination precede and underlie the discursive concepts of the understanding. Hence Kant regards imagination as a genuine principle of objectification; in this sense he explicitly qualifies the schemata as "realizing," conditioning the object and rendering it possible.

Modern psychology of perception has presented this concept of the objectifying and realizing schema in a new light. In modern psychology it appears clearly that there exists a peculiar function to which perception owes its objectivity. The "true" color, the "true" shape, the "true" size of an object are by no means that which is given in any particular impression, nor need they be the "sum" of these impressions. For a satisfactory account, the function of memory, the reference to reproductive processes, are not sufficient either. The *constitutive* factor must be sought somewhere else; this factor manifests itself in the possibility of forming invariants. Owing to this possibility, there exist for us a "perspective of illumination" and a spatial perspective and thus the perception of "objective" reality. The factor of organization possesses, then, positive, not merely negative, significance. Hering, as we saw, explains the *significance* of "perceptual constancy" by the fact that objective knowledge and objective judgment are rendered possible by this constancy. If there were no such constancy, we would, as it were, abandon ourselves to every change in external conditions; it would be impossible to segregate "things" and "properties" from the stream of becoming. To use Heraclitus' metaphore, we should in fact be unable to "step down twice into the same river." A piece of chalk, as Hering shows, would, on a cloudy day, present the same color as a piece of coal on a sunshiny day, and in the course of one day it would display all possible colors intermediate between black and white. "A white flower seen under green foliage would display the same color as a green leaf of a tree in the open air, and a ball of thread, white in daylight, must, in gas-light, have the color of an orange."[65] Thus psychology, as compared to its early sensationalistic beginnings, has achieved a thoroughgoing revaluation. Psychology dismisses the dogma of the strict one-to-one correspondence between physical stimuli and perceptions. It is, on the contrary, the "transformed" impression, i.e., the impression as modified with respect to the various phenomena of constancy, which is regarded as the "true" impression, since we can on these grounds construct knowledge of reality. This, it seems to me, is a momentous step; for in no other way could the traditional separation, and even opposition, between the "psychological" and the "epistemological" problem in the domain of perception be overcome. On this new basis psychology and epistemology may meet and cooperatively attack the numerous problems still to be solved.*

ERNST CASSIRER.

COLUMBIA UNIVERSITY.

EXTRACTO

Este ensayo ofrece un análisis del concepto de objetividad en función del concepto de invariabilidad con relación a un grupo especial de transformaciones, a fin de revelar las analogías existentes entre los fenómenos de constancia perceptibles y las "propiedades" geométricas tal como se definen en la teoría de los grupos. De acuerdo con el principio de la teoría de los grupos, sólo pueden considerarse como geométricas y "objetivas" aquellas propiedades que permanecen invariables a través del tipo de transformaciones definido por el "grupo" que sustenta el sistema geométrico correspondiente. Así pues, las diversas cónicas, que son conceptualmente *distintas* con relación al grupo definido por la geometría euclidiana, son conceptualmente *idénticas* dentro de un sistema cuyo grupo definidor contenga transformaciones proyectivas o afines. Esta función de "objetivación" por medio de la selección de ciertas reglas de transformación, aparece en forma rudimentaria aun en el dominio de la "pura" percepción. Las investigaciones psicológicas modernas han puesto de manifiesto la inadecuación de la tradicional concepción sensacionista de la percepción, que considera a ésta como un proceso de reproducción fotográfica, o como el reflejo de estímulos atómicos "dados." Está probado experimentalmente que, en lenguaje kantiano, la experiencia *posible* (percepción) entra como un factor constitutivo en la experiencia *real* (percepción). Luego,

[65] Hering, *Grundzuege*, p. 16.
* Translation by Aron Gurwitsch.

y puesto que las condiciones de iluminación de nuestras perspectivas espaciales varían siempre, no sería posible percibir el "verdadero" color o el tamaño de un objeto sin operar la selección de propiedades relativamente invariables. Como la psicología de la forma ha mostrado, nosotros percibimos "formas" (por ejemplo, melodías) que gozan de cierta independencia respecto de las "materias" (en el caso de una melodía serían las notas, por lo que se refiere a sus posiciones absolutas) en que aquellas están incorporadas en el momento transitorio de ser percibidas. Esta percepción de invariables no puede ser explicada en función de hipótesis dudosas como la de la memoria racial o "inferencia inconsciente" (Helmholtz), sino que debe ser admitida como una innata función de "objetivación."